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Lorentz's Pendulum Problem*

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1. At the first Solvay Congress in 1911, Lorentz raised the question: How does a simple pendulum behave when the suspending thread is gradually shortened? This was relevant to the quantum theory of that date.

The problem may be presented as follows. In the equation

$$\ddot{x} + \omega^2 x = 0, \qquad y = \dot{x} \quad (-\infty < t < \infty), \quad \text{(D)}$$

we suppose that

$$\omega \geqq b_0 > 0; \qquad |\omega^{(n)}| \leqq b_n \epsilon^n, \quad \int_{-\infty}^{\infty} |\omega^{(n)}|\, dt \leqq b_n' \epsilon^{n-1} \qquad (n \geqq 1).$$

$$\omega \to \omega(\pm\infty), \qquad \omega^{(n)} \to 0 \ (n \geqq 1) \qquad \text{as} \qquad t \to \pm\infty.$$

These would in particular be satisfied if

$$\omega(t) = \varphi(\epsilon t), \qquad \text{where} \quad |\varphi^{(n)}(x)| \leqq b_n \qquad (-\omega < x < \infty), \text{etc.}$$

Let

$$H(t) = (\omega^2 x^2 + y^2)/\omega = \omega x^2 + \omega^{-1} y^2.$$

It has been conjectured that H is approximately constant; that is to say, the energy $E = \omega^2 x^2 + y^2$ is approximately proportional to ω; a further conjecture is the very drastic

$$H(\infty) - H(-\infty) = O(\epsilon^n)$$

for every n. Various heuristic arguments have appeared, but no rigorous proof of anything for the range $-\infty < t < +\infty$; such results are known for a $\varphi(x)$ constant when $x < 0$ and when $x > 1$, and with $\varphi^{(n)}(0) = \varphi^{(n)}(1) = 0$ for all n.

I shall prove:

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THEOREM 1.
(i) $H(t) = H(\infty) - \dot\omega\omega^{-2}xy + \frac{1}{2}(\ddot\omega\omega^{-4} - 2\dot\omega^2\omega^{-5})(y^2 - \omega^2x^2) - \frac{1}{8}\dot\omega^2\omega^{-4}H(\infty) + O(\epsilon^2).$

(ii) *Let*

$$x = (H\omega^{-1})^{1/2}\cos\alpha, \qquad y = -(H\omega)^{1/2}\sin\alpha$$

at time t. Let $p = 2\pi/\omega(t)$ *be the "local period." Then for the average* $\mathfrak{A}_p(H)$ *of H over* $(t - \frac{1}{2}p, t + \frac{1}{2}p)$ *we have*

$$\mathfrak{A}_\alpha(H(t)) = \frac{1}{p}\int_{t-\frac{1}{2}p}^{t+\frac{1}{2}p} H(\tau)\,d\tau = H(\infty) + O(\epsilon^2).$$

(iii) $\mathfrak{A}_\alpha\{H(t)\} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} H(t)\,d\alpha = -\frac{1}{8}\dot v^2 H(\infty) + O(\epsilon^3)\left(-\frac{1}{2}p \leqq t \leqq \frac{1}{2}p\right).$

(iv) $H(\infty) - H(-\infty) = O(\epsilon^n)$ *for every n.*

We observe (a) that the error in $H(t)$ is of exact order ϵ; (b) that, by (ii) and (iii), this is improved to exact order ϵ^2 by averaging over a period; (c) that no further improvement results from a further averaging over the initial phase α.

We will use B's to denote positive constants depending only on the b_n and b_n': B's are not in general the same from one occurrence to the next.

2. LEMMA 1. $B < H(t_2)/H(t_1) < B$ *for all* t_1, t_2.

We have

$$\dot H = |\,\dot\omega(x^2 - y^2/\omega^2)\,| \leqq BH\dot\omega,$$

$$|\log\{H(t_2)/H(t_1)\}| \leqq \exp\left(\left|\int_{t_1}^{t_2} B\,|\,\dot\omega\,dt\right|\right)$$

$$\leqq \exp\left(B\int_{-\infty}^{\infty}|\,\dot\omega\,|\,dt\right) < \exp B = B,$$

which is equivalent to the result of the Lemma.

As a corollary we may note the result, of some interest in itself:

THEOREM 2. *If* $P_0 = (x_0, y_0)$ *and* $P_1 = (x_1, y_1)$ *represent two sets of initial conditions at* $t = t_0$, *and if* P_0, P_1 *are near (distance* $< \delta$*), then* $P_0(t)$, $P_1(t)$, *the corresponding* (x, y)*'s at any time t are also near (distance* $< B\delta$*).*

After Lemma 1 we use O's to imply constants depending only on the b_n and b_n' and $H(0)$ (or presently, when the existence of $H(\infty)$ is proved, on $H(\infty)$). (An $O(1)$ is generally $< BH(0)$ or $< B\sqrt{H(0)}$.)

3. At $t = 0$ let

$$x_0 = a\cos\alpha, \qquad y_0 = -b\sin\alpha, \qquad a = (H_0\omega_0^{-1})^{1/2}, \qquad b = (H_0\omega_0)^{1/2}.$$

The solution of

$$\ddot{X} + \omega_0^2 X = 0, \tag{1}$$

with initial values x_0, y_0 is

$$X = a\cos(\omega_0 t + \alpha), \qquad Y = -b\sin(\omega_0 t + \alpha).$$

The solution (x, y) of (D) with initial values x_0, y_0 is

$$x = X + \xi, \qquad y = Y + \eta, \qquad Y = \dot{X}, \qquad \eta = \dot{\xi}, \tag{2}$$

where

$$\xi = \frac{1}{\omega_0}\int_0^t \{X(\tau) + \xi(\tau)\}\{\omega^2(\tau) - \omega_0^2\}\sin\omega_0(t - \tau)\,d\tau. \tag{3}$$

We shall not be using ξ, η except in a range of t of length $<B$; over any such range $\omega^2(\tau) - \omega_0^2$ is $O(\epsilon)$. Hence

$$\xi = \xi_1 + O(\epsilon^2), \qquad \xi_1 = \frac{1}{\omega_0}\int_0^t X(\tau)\{\omega^2(\tau) - \omega_0^2\}\sin\omega_0(t - \tau)\,d\tau; \tag{4}$$

$$\eta = \eta_1 + O(\epsilon^2), \qquad \eta_1 = \int_0^t X(\tau)\{\omega^2(\tau) - \omega_0^2\}\cos\omega_0(t - \tau)\,d\tau; \tag{5}$$

$$\xi, \eta = O(1)\int_0^t |\,\omega(\tau) - \omega_0\,|\,dt + O(\epsilon^2) = O(\epsilon). \tag{6}$$

4. Lemma 2.

$$\int_{t_0}^t xy\,dt = O(1) \qquad (\textit{all } t_0, t).$$

Let $t_1 = t_0 + 2\pi/\omega(t_0)$, $t_2 = t_1 + 2\pi/\omega(t_1)$, $\cdots$. Supposing (as we may) that $t > t_0$, let t_n be the last $t_m \leqq t_1$. Then

$$\int_{t_0}^t xy\,dt = \sum^{n-1}\int_{t_m}^{t_{m+1}} xy\,dt + O(1).$$

In (t_0, t_1), representative of the general (t_m, t_{m+1}), and supposing that $t_0 = 0$, we have

$$\begin{aligned}\int_{t_0}^{t_1} xy\,dt &= \int_0^{t_1}\{XY + O(\xi) + O(\dot{\xi})\}\,dt\\ &= 0 + O(1)\int_0^{t_1} dt\int_0^t |\,(\omega(\tau) - \omega_0)\,|\,d\tau\\ &= O(1)\int_0^{t_1} dt\left\{\int_0^t\left(\int_0^\tau |\,\dot{\omega}(u)\,|\,du\right)d\tau\right\}\\ &\leqq O(1)\int_0^{t_1} du B\,|\,\dot{\omega}(u)\,|,\end{aligned}$$

on changing the order of integration in the triple integral to $d\tau\, dt\, du$. Hence

$$\left| \int_{t_0}^{t} xy\, dt \right| < O(1) + O(1)\cdot\sum \int_{t_m}^{t_{m+1}} |\dot\omega(u)|\, du$$

$$< O(1) + O(1) \int_{-\infty}^{\infty} |\dot\omega(t)|\, dt = O(1),$$

as desired.

5. Abbreviate xy to U and ω^{-1} to v (negative powers of ω occur freely). We have

$$\dot H = \dot\omega(x^2 - y^2/\omega^2), \qquad \dot U = y^2 - \omega^2 x^2$$

$$H(T) - H(t) = \int_t^T \dot v \dot U\, dt = [\dot v U]_t^T - \int_t^T \ddot v U\, dt. \tag{7}$$

Let $T \to \infty$, when $\dot v(T) \to 0$, $\int_T^\infty |\ddot v U|\, dt \leqq B\int_T^\infty |\ddot v|\, dt \to 0$, since $\int_{-\infty}^{\infty}$ exists. Hence $H(\infty) = \lim_{T\to\infty} H(t)$ exists, and

$$H(\infty) - H(t) = -\dot v U - \int_t^\infty \ddot v U\, dt \tag{8}$$

Now it is easily verified that

$$\begin{aligned} U &= -\tfrac{1}{4}\omega^{-2}\ddot U + \tfrac{1}{4}\dot\omega\omega^{-3}\dot U - \tfrac{1}{4}\dot\omega\omega^{-2} H \\ &= -\tfrac{1}{4}v^2\ddot U - \tfrac{1}{4}v\dot v\dot U + \tfrac{1}{4}\dot v H. \end{aligned} \tag{9}$$

Hence

$$\begin{aligned} \int_t^\infty \ddot v U\, dt &= \left[\frac{1}{4} v^2 \ddot v \dot U\right]_t^\infty + \frac{1}{4}\int_t^\infty \frac{d}{dt}(v^2\ddot v)\dot U\, dt \\ &\quad -\tfrac{1}{4}\int_t^\infty v\dot v\ddot v\dot U\, dt + \tfrac{1}{4}\int_t^\infty \dot v\ddot v H\, dt \\ &= -\tfrac{1}{4}v^2\ddot v\dot U + \tfrac{1}{4}\int_t^\infty \dot U\{2v\dot v\ddot v + v^2\dddot v - v\dot v\ddot v\}\, dt \\ &\quad -\tfrac{1}{8}\dot v^2 H(\infty) + O(1)\int_{-\infty}^\infty |\dot v\ddot v|\,\epsilon\, dt. \end{aligned} \tag{10}$$

Integrating the second term by parts shows that it is of the form

$$O(1)\{\ \} + O(1)\int_{-\infty}^\infty \left|\frac{d}{dt}\{\ \}\right| dt = O(\epsilon^3) + O(\epsilon^3) = O(\epsilon^3).$$

The fourth term is also $O(\epsilon^3)$, and we find

$$\int_t^\infty \ddot v U\, dt = -\tfrac{1}{4}v^2\ddot v(y^2 - \omega^2 x^2) - \tfrac{1}{8}\dot v^2 H(\infty) + O(\epsilon^3).$$

Hence, from (8),

$$H(t) - H(\infty) = \dot{v}xy - \tfrac{1}{4}v^2\ddot{v}(y^2 - \omega^2x^2) + \tfrac{1}{8}\dot{v}^2H(\infty) + O(\epsilon^3), \quad (11)$$

and this is equivalent to (i) of Theorem 1.

6. We take up next the question of

$$\mathfrak{A}_p\{H(t)\} = \frac{1}{p}\int_{t-1/2p}^{t+1/2p} H(\tau)\,d\tau, \qquad p = \frac{2\pi}{\omega(t)}.$$

We may suppose $t = 0$.

By Eq. (10)

$$\begin{aligned}\frac{1}{p}\int_{-1/2p}^{1/2p} H(t)\,dt - H(\infty) &= \frac{1}{p}\int_{-1/2p}^{1/2p} \dot{v}U\,dt \\ &\quad - \frac{1}{4p}\int_{-1/2p}^{1/2p} v^2\ddot{v}(y^2 - \omega^2x^2)\,dt + \frac{1}{8p}H(\infty)\int_{-1/2p}^{1/2p} \dot{v}\,dt + O(\epsilon^3) \quad (11) \\ &= \frac{1}{p}\int_{-1/2p}^{1/2p} \dot{v}U\,dt + O(\epsilon^2).\end{aligned}$$

Now by (9)

$$\begin{aligned}\int_{-1/2p}^{1/2p} \dot{v}U\,dt &= -\tfrac{1}{4}\int_{-1/2p}^{1/2p} \dot{v}v^2\ddot{U}\,dt + \int_{-1/2p}^{1/2p} O(\epsilon^2)\,dt \\ &= \left\{-\tfrac{1}{4}[\dot{v}v^2\dot{U}]_{-1/2p}^{1/2p} + \int_{-1/2p}^{1/2p} O(\epsilon^2)\right\} + O(\epsilon^2) = O(\epsilon^2),\end{aligned}$$

since $\dot{U}(\tfrac{1}{2}p) - \dot{U}(-\tfrac{1}{2}p) = O(\epsilon)$ and since $\dot{v}v^2$ varies by $O(\epsilon^2)$ over $(-\tfrac{1}{2}p, \tfrac{1}{2}p)$. Hence, $\mathfrak{A}_p\{H(t)\} = H(\infty) + O(\epsilon^2)$, which is (ii) of Theorem 1.

7. Consider now $\mathfrak{A}_\alpha\{H(t)\}$. We may again suppose the t here is 0, and that the t's in what follows satisfy $-\tfrac{1}{2}p \leqq t \leqq \tfrac{1}{2}p$. By (8)

$$\mathfrak{A}_\alpha\{H(t)\} - H(\infty) = \mathfrak{A}_\alpha(\dot{v}U) + \mathfrak{A}_\alpha\int_t^\infty \ddot{v}U\,dt + O(\epsilon^3). \quad (12)$$

Now by (10),

$$\begin{aligned}\int_t^\infty \ddot{v}U\,dt = -\frac{1}{4}v^2\ddot{v}\dot{U} - \frac{1}{4}\frac{d}{dt}(v^2\ddot{v})U - \frac{1}{4}\int_t^\infty \frac{d^2}{dt^2}(v^2\ddot{v}U)\,dt \\ + \frac{1}{4}(v\dot{v}\ddot{v})U + \frac{1}{4}\int_t^\infty \frac{d}{dt}(v\dot{v}\ddot{v})U\,dt \\ -\tfrac{1}{8}\ddot{v}^2H(\infty) + \tfrac{1}{4}\int_t^\infty \dot{v}\ddot{v}\{H - H(\infty)\}\,dt.\end{aligned}$$

The three integrals on the right are each $O(\epsilon^3)$, and since $\mathfrak{A}_\alpha(U) = O(\epsilon)$, $\mathfrak{A}_\alpha(\dot{U}) = O(\epsilon)$, we have

$$\mathfrak{A}_\alpha\left(\int_t^\infty \ddot{v} U\, dt\right) = -\tfrac{1}{8}H(\infty)\dot{v}^2 + O(\epsilon^3). \tag{13}$$

Next,

$$\begin{aligned}\mathfrak{A}_\alpha(\dot{v}U) &= \mathfrak{A}_\alpha\{\dot{v}(xy + x\eta_1 + Y\xi_1)\} + O(\epsilon^3)\\ &= \dot{v}\mathfrak{A}(X\eta_1 + Y\xi_1) + O(\epsilon^3), \quad \text{since} \quad \mathfrak{A}_\alpha(XY) = 0,\\ &= \dot{v}\mathfrak{A}_\alpha\left\{\frac{d}{dt}(X\xi_1)\right\} + O(\epsilon^3).\end{aligned}$$

Now $\omega^2(\tau) - \omega_0^2 = 2\omega_0\dot{\omega}_0\tau + O(\epsilon^2)$ for a bounded range of τ. Substituting this and $X(\tau) = a\cos(\omega_0\tau + \alpha), Y = -a\sin(\omega_0\tau + \alpha)$ in (4) and (5), we have, after some rearrangement,

$$\begin{aligned}\xi_1 = a\dot{\omega}_0 \sin(\omega_0 t + \alpha) &\int_0^t \tau\{1 + \cos 2(\omega_0\tau + \alpha)\}\, d\tau\\ - a\dot{\omega}_0\cos(\omega_0 t + \alpha) &\int_0^t \tau \sin 2(\omega_0\tau + \alpha)\, d\tau + O(\epsilon^2),\end{aligned}$$

$\eta_1 =$ the formal derivative of this,

$$\begin{aligned}\frac{d}{dt}\frac{X\xi_1}{a^2\dot{\omega}_0} &= \frac{d}{dt}\int_0^t \tau[\tfrac{1}{2}\sin 2(\omega_0 t + \alpha) + \tfrac{1}{4}\sin\{2\omega_0(t + \tau) + 4\alpha\}\\ &\quad + \tfrac{1}{4}\sin 2\omega_0(t - \tau)]\, d\tau - \frac{d}{dt}\int_0^t \tau[\tfrac{1}{2}\sin 2(\omega_0\tau + \alpha)\\ &\quad + \tfrac{1}{4}\sin\{2\omega_0(t + \tau) + 4\alpha\} - \tfrac{1}{4}\sin 2\omega_0(t - \tau)]\, d\tau + O(\epsilon^2).\end{aligned}$$

The last terms in the two square brackets cancel, and $\mathfrak{A}_\alpha$ (taken under the integral sign) annihilates all other terms. So

$$\mathfrak{A}(X\xi_1) = O(\epsilon^3).$$

Combining this with (12) and (13) we get (iii) of Theorem 1.

8. We come now to the last and most striking part of Theorem 1. By (7)

$$\Delta = H(\infty) - H(-\infty) = [\dot{v}U]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \ddot{v}U\, dt = -\int_{-\infty}^{\infty} v_2 U\, dt, \tag{14}$$

where, as throughout this section, we use suffixes to denote differentiation (high numbers are involved). Now, from (9),

$$U = -\tfrac{1}{4}v^2 U_2 - \tfrac{1}{4}vv_1 U_1 + \tfrac{1}{4}v_1 H. \tag{15}$$

So, from (14),

$$4\Delta = \int_{-\infty}^{\infty} v_2(v^2 U_2 + vv_1 U_1 - v_1 H)\, dt. \tag{16}$$

Integrate twice by parts for the term in U_2, and once for that in U_1; the integrated terms vanish at the limits. Finally, since

$$\int_{-\infty}^{\infty} v_1 v_2 = [\tfrac{1}{2}v_1^2]_{-\infty}^{\infty} = 0,$$

we may replace H in (16) by $H - H(\infty)$. We thus obtain

$$4\Delta = \int_{-\infty}^{\infty} \{(v^2 v_2)_2 - (vv_1 v_2)_1\} U\, dt - \int_{-\infty}^{\infty} v_1 v_2 \{H - H(\infty)\}\, dt. \tag{17}$$

Since

$$H(t) - H(\infty) = -\int_t^{\infty} v_1 U_1\, dt,$$

by (7), and since

$$\int_{-\infty}^{\infty} \left\{ v_1 v_2 \int_{-t}^{\infty} v_1 U_1\, dt \right\} dt = \left[\tfrac{1}{2}v_1^2 \int_t^{\infty} v_1 U_1\, dt \right]_{-\infty}^{\infty} + \tfrac{1}{2} \int_{-\infty}^{\infty} v_1^3 U_1\, dt,$$

$$= 0 + [\tfrac{1}{2}v_1^3 U]_{-\infty}^{\infty} - \tfrac{3}{2} \int_{-\infty}^{\infty} v_1^2 v_2 U\, dt = -\tfrac{3}{2} \int_{-\infty}^{\infty} v_1^2 v_2 U\, dt,$$

we have from (17),

$$4\Delta = \int_{-\infty}^{\infty} VU\, dt, \qquad V = V^{(1)} = (v^2 v_2)_2 - (vv_1 v_2)_1 + \tfrac{3}{2}v_1^2 v_2. \tag{18}$$

We obtained this by starting with

$$-\Delta = \int_{-\infty}^{\infty} V^{(0)} U\, dt, \qquad V^{(0)} = v_2. \tag{19}$$

If now we repeat the process, *mutatis mutandis*, with $V^{(1)} = V$ in place of $V^{(0)}$, we find

$$-4^2\Delta = \int_{-\infty}^{\infty} \{(v^2 V)_2 - (vv_1 V)_1\}\, dt - \int_{-\infty}^{\infty} v_1 VH\, dt. \tag{20}$$

If now $v_1 V$ is a perfect differential W_1, we may replace H by $H - H(\infty)$, and the former procedure leads to

$$-4^2\Delta = \int_{-\infty}^{\infty} V^{(2)} U\, dt, \qquad V^{(2)} = (v^2 V^{(1)})_2 - (vv_1 V^{(1)})_1 - (Wv_1)_1. \tag{21}$$

Now $v_1 V = v_1(v^2 v_2)_2 - v_1(vv_1v_2)_1 + \tfrac{3}{2}v_1^3 v_2 = W_1$, where

$$W = v_1(v^2v_2)_1 - vv_1^2v_2 - \tfrac{1}{2}v^2v_2^2 + \tfrac{3}{8}v_1^4,$$

and we have (21) with this value of W.

The process can be continued so long as $v_1 V^{(n)}$ is a perfect differential $W_1^{(n)}$, and in fact it always is one. If $v_1 V^{(n)}$ is not a $W_1^{(n)}$ we should have in the analogue of (20) a term $\pm 4^{-(n+2)} \int_{-\infty}^{\infty} v_1 V^{(n)}\, dt$, which would in general be of exact order ϵ^{2n}, contrary to (iii) of Theorem 1 (which we shall prove). But there does not seem to be an inductive argument for $v_1 V^{(n)} = W_1^{(n)}$: the result depends on the particular value of $V^{(0)}$. Fortunately there is no need to struggle with the identity, for another line is available.

9. In what follows we stop using suffixes for differentiation; they will have other uses.

An obvious transformation for (D) is $t_1 = \omega\, dt$, when (D) becomes

$$\frac{d^2x}{dt_1^2} + x = -\left(\frac{1}{\omega}\frac{d\omega}{dt_1}\right)\frac{dx}{dt_1}.$$

Chandrasekhar (*1*) introduced the further transformation by $x = \omega^{1/2}x$, (which is what removes the term in dx/dt_1). This leads to

$$\frac{d^2x}{dt_1^2} + \omega_1^2 x_1 = 0, \qquad \omega_1^2 = 1 + \Omega_1, \qquad \Omega_1 = \omega^{-3/2}\left(\frac{d}{dt}\right)^2 (\omega^{-1/2}). \qquad \text{(D)}_1$$

This equation is remarkable in that, not only do we have $\omega_1 = 1 + O(\epsilon^2)$, but the transformation can be iterated.

Now it is not the case that

$$H(t) = H_1(t_1) = \omega_1 x_1^2 + y_1^2/\omega_1 .$$

We have, in fact,

$$y_1 = \frac{dx_1}{dt_1} = \omega^{-1/2} y + \tfrac{1}{2}\omega^{-3/2}\dot\omega x,$$
$$H_1(t_1) = \omega\omega_1 x^2 + \omega_1^{-1}(\omega^{-1}y^2 + \omega^{-2}\dot\omega xy + \tfrac{1}{4}\omega^{-3}\dot\omega^2 x^2). \qquad (22)$$

But since $\omega_1(\pm\infty) = 1$ and $\dot\omega(\pm\infty) = 0$ we do have from (22)

$$H(\infty) = H_1(\infty), \qquad H(-\infty) = H_1(-\infty), \qquad \Delta = \Delta_1 . \qquad (23)$$

We now iterate the transformation:

$$dt_{n-1} = \omega_n\, dt_n , \qquad x_{n+1} = \omega_n^{1/2} x_n ,$$

and observe that, by (9), in which the $O(\epsilon^2)$ becomes 0 at ∞,

$$|\Delta| = |\Delta_n| \leqq \beta H(\infty) \int_{-\infty}^{\infty} \left| \left(\frac{d}{dt_n}\right)^3 \Omega_n \right| dt_n , \qquad (24)$$

where β, a certain B, is the constant in

$$\operatorname{Max}\left|\int_0^t U_n\,dt\right| \leqq \beta H(\infty) \tag{25}$$

(the O in Lemma 2 implies (25) with β a B). Since $dt_n = \omega\cdot\omega_1\omega_2\cdots\omega_{n-1}\,dt = \omega\,dt\,(1 + \text{higher terms})$, and since

$$\Omega_n = \left(\frac{d}{dt_{n-1}}\right)^2 (1+\Omega_{n-1})^{-1/4} = \left\{-\frac{1}{4}\left(\frac{d}{dt_1}\right)^2 \Omega_{n-1}\right\}(1 + \text{higher terms})$$

we conclude from (24) that

$$|\Delta| = |\Delta_n| \leqq \beta\{1 + O(\epsilon^2)\}2^{-2n}\int_{-\infty}^{\infty}\left|\left(\frac{d}{dt}\right)^3\left(\frac{d}{\omega\,dt}\right)^{2n-2}\right|dt.$$

This is $O(\epsilon^{2n+2})$.

I am indebted to Professor S. Chandrasekhar for bringing the problem to my notice, and for numerous discussions.

ADDENDUM

10. [Added December 14, 1962. The work was done under Contract Nonr 222(60).] I can now prove Theorem 1(iv) without using Chandrasekhar's transformation. Such a proof is very desirable in view of other problems of Adiabatic Invariance over infinite time, where nothing corresponding to the transformation is available.

In arriving at (14) and (18) of §8 we integrated by parts a number of times, the integrated parts vanishing at $-\infty$. If we were dealing with $H(\infty) - H(t)$ instead of Δ we should have obtained, for the general analogue of (7), §8,

$$H(\infty) - H(t) = Q_n(t) - 4^{-n}\int_t^{\infty} V^{(n)}U\,dt - 4^{-n}\int_t^{\infty} v_1 V^{(n-1)}H(t)\,dt,$$

$$H(t) = H(\infty) - Q_n + 4^{-n}\int_t^{\infty}\left(V^{(n)}U + v_1 V^{(n-1)}\{H - H(\infty)\}\right)dt \tag{26}$$

$$+ 4^{-n}H(\infty)\int_t^{\infty} v_1 V^{(n-1)}\,dt.$$

In this, $Q_n = a_n(t)x^2 + 2b_n(t)xy + c_n(t)y^2$, Q_n vanishes at $-\infty$, and $V^{(n)}$ has weight $2n + 2$ in differentiations, so that

$$\int_{-\infty}^{\infty}|V^{(n)}U|\,dt = O(\epsilon^{2n+1}), \qquad \int_{-\infty}^{\infty}|V^{(n-1)}\{H - H(\infty)\}|\,dt = O(\epsilon^{2n+1}).$$

We have seen that (26) is valid for $n = 0, 1$, and that in each case $v_1 V^{(n-1)}$ is a perfect differential,

$$v_1 V^{(n-1)} = \frac{dw}{dt}. \tag{27}$$

Let now n be the first, if any, n for which (27) is false. Then there will exist a function $\omega(t)$, satisfying (H), for which

$$4^{-n} \int_{-\infty}^{\infty} v_1 V^{(n-1)}\, dt = K_n \epsilon^{2n}, \qquad \text{where} \quad |k_n| > b_n^* > 0.$$

Consider the function H for this particular ω. Let $C(\infty)$ be the curve in (h, k) space:

$$H(h, k; \infty) = \omega(\infty)h^2 + \omega^{-1}(\infty)k^2 = 1, \tag{28}$$

say. Let (x, y) be the solution of (D) at time t corresponding to (h, k) at $t = \infty$, and let $C(t)$ be the curve corresponding to (27) at time t. $C(t)$ has the equation

$$\begin{aligned} \omega x^2 + w^{-1}y^2 = H(x, y; t) = H(h, k; \infty) - Q_n \\ + 4^{-n} \int_t^{\infty} (V^{(n)}U + v_1 V^{(n-1)}\{H(t) - H(\infty)\})\, dt \\ + 4^{-n}H(x, y; \infty) \int_t^{\infty} v_1 V^{(n-1)}\, dt, \end{aligned} \tag{29}$$

$$\omega x^2 + \omega^{-1}y^2 = 1 - a_n(t)x^2 - 2b_n(t)xy - c_n(t)y^2 + O(\epsilon^{2n+1}) + k_n \epsilon^{2n},$$

since $H(x, y; \infty) = H(h, k; \infty) + O(\epsilon) = 1 + O(\epsilon)$.

Now, by Liouville's Theorem [valid for the special equations (D)], the area contained in the curve (29) is equal to that contained in (28), namely $\pi + O(\epsilon^{2n+1})$. Now (29) for $t = -\infty$ is

$$\omega(-\infty)x^2 + \omega^{-1}(-\infty)(y^2) = 1 + O(\epsilon^{2n+1}) + k_n\epsilon^{2n},$$

and its area is $\pi\{1 + O(\epsilon^{2n+1}) + k_n \epsilon^n\}$, which differs from $\pi + O(\epsilon^{2n+1})$. Hence we have always $v^1V^{(n-1)}\, dt = dw/dt$, and our task is completed.

I mention in conclusion that I have solved the corresponding problem of approximate elliptic motion about a central force $-\omega r^{-2}$ with ω varying as in the present problem. The results are parallel. e^2 is an Adiabatic Invariant. Assuming $0 < b < e(0) < 1 - b$, e^2 is constant to error $O(\epsilon)$ exactly, its average over a period is constant to $O(\epsilon^2)$ exactly, and

$$e^2(\infty) - e^2(-\infty) = O(\epsilon^n).$$

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